

Quantum Field Theory

Set 5

Exercise 1: The Lie theorem

Consider a general Lie group $\mathcal{G}$ parametrized in a neighborhood of the identity by some choice of coordinates $\alpha \in \mathbb{R}^n$ so that the origin parametrizes the identity element $g(0) = e$. Given two group elements $g(\alpha)$ and $g(\beta)$, we can define a function $p(\alpha, \beta)$ that implements the product rule of the group

$$g(\alpha)g(\beta) = g(p(\alpha, \beta)).$$

The manifold structure of the Lie group tells us that this function is smooth, and in particular it can be expanded in series in a finite neighborhood of the identity.

- Using the group properties compute $p(\alpha, 0)$ and $p(0, \beta)$.
- Using the previous result, write the most general expansion of $p(\alpha, \beta)$ up to the second order in α and β (included).
- Show that the symmetric part of the coefficient of the second order in the expansion can be eliminated by a suitable change of parametrization $\alpha^i \rightarrow \alpha^i + \delta_{ab}^i \alpha^a \alpha^b + O(\alpha^3)$, while the antisymmetric part can't.
- Call $\bar{\alpha}$ the coordinate of the inverse element of α (in formulas $g(\alpha)^{-1} = g(\bar{\alpha})$). Find an expression for $\bar{\alpha}(\alpha)$ up to the second order in α (again included).
- Consider the following product $g(\alpha)^{-1}g(\beta)^{-1}g(\alpha)g(\beta) \equiv g(c(\alpha, \beta))$. Compute $c(\alpha, \beta)$ up to the second order in α and β and show that it satisfies the antisymmetry property of the Lie product.
- **Bonus (hard)**: use the associativity of the product to show that the Lie product satisfies the Jacobi identity.
Hint: you will have to go to the third order in the expansion.

Exercise 2: Irreducible representations of $SU(2)$

Consider the Algebra of the group $SU(2)$

$$[T^a, T^b] = i\epsilon^{abc}T^c.$$

- Find all the commutation rules between the following quantities

$$T^\pm = \frac{T^1 \pm iT^2}{\sqrt{2}}; \quad T^3.$$

- Show that the *Casimir operator* $J^2 \equiv (T^1)^2 + (T^2)^2 + (T^3)^2$ is proportional to the identity in any irreducible representation: $(\tau)^2 = \mu^2 \times 1_N$. (Here we denote as $(\tau)^2$ the representative of the operator J^2).

Consider an irreducible representation of the Algebra on a given vector space V , endowed with a usual scalar product satisfying $\langle n|n \rangle \equiv ||n||^2 \geq 0$. Denote $|m\rangle$ an eigenvector of the generator τ^3 relative to the eigenvalue m .

$$\tau^3|m\rangle = m|m\rangle.$$

- Compute the action of $\tau^\pm$ on $|m\rangle$.
- Prove that $|m| + m^2 \leq \mu^2$.
- Construct the irreducible representation which $|m\rangle$ belongs to.
- Compute the dimension of the representation and show that $\mu^2 = j(j+1)$ with j integer or semi-integer.
- Construct the representation corresponding to $j = 1/2$ and $j = 1$.

Exercise 3: Direct sum and tensor product

Given two finite-dimensional inequivalent irreducible representations D_1 and D_2 of a group G on two vector spaces V_1 and V_2 ($D_1 \in GL(V_1)$ and $D_2 \in GL(V_2)$ where $GL(V)$ is the set of all linear transformations from V to itself):

- Show that the operators $D(g) \in GL(V_1 \oplus V_2)$, $g \in G$ defined by

$$D(g)(v_1 \oplus v_2) \equiv D(g)(v_1, v_2) = (D_1(g)v_1, D_2(g)v_2), \quad v_1 \in V_1, v_2 \in V_2, \quad (1)$$

furnish a (reducible) representation of G (called the direct sum of D_1 and D_2). What is the dimension of the vector space $V_1 \oplus V_2$?

Given an operator $A \in GL(V_1 \oplus V_2)$ and $A \neq 0$, such that $AD(g) = D(g)A$ for every $g \in G$ show that $A(v_1 \oplus v_2)^T = (\lambda_1 v_1, \lambda_2 v_2)^T$, that is A is a multiple of the identity over both subspaces $(V_1, 0)$ and $(0, V_2)$ of $V_1 \oplus V_2$.

- Show that the operators $D(g) \in GL(V_1 \otimes V_2)$, $g \in G$ defined by

$$D(g)(v_1 \otimes v_2) = D_1(g)v_1 \otimes D_2(g)v_2, \quad v_1 \in V_1, v_2 \in V_2, \quad (2)$$

where $\otimes$ is associative and distributive (like the usual product), furnish a representation of G (called the direct product of D_1 and D_2). What is the dimension of the vector space $V_1 \otimes V_2$?

Express the generators of D in terms of the generators of D_1 and D_2 . Show that they ‘add’.